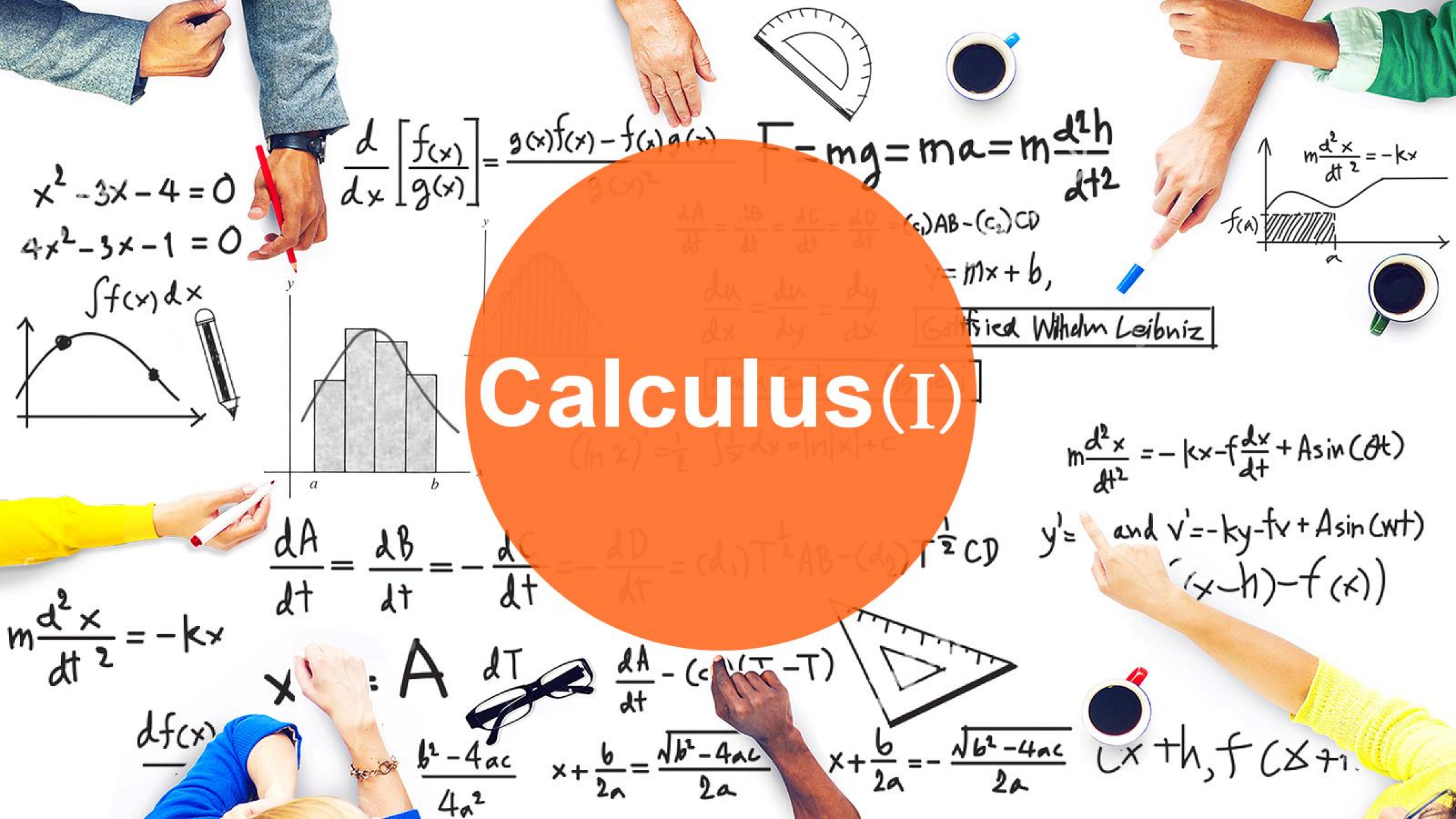


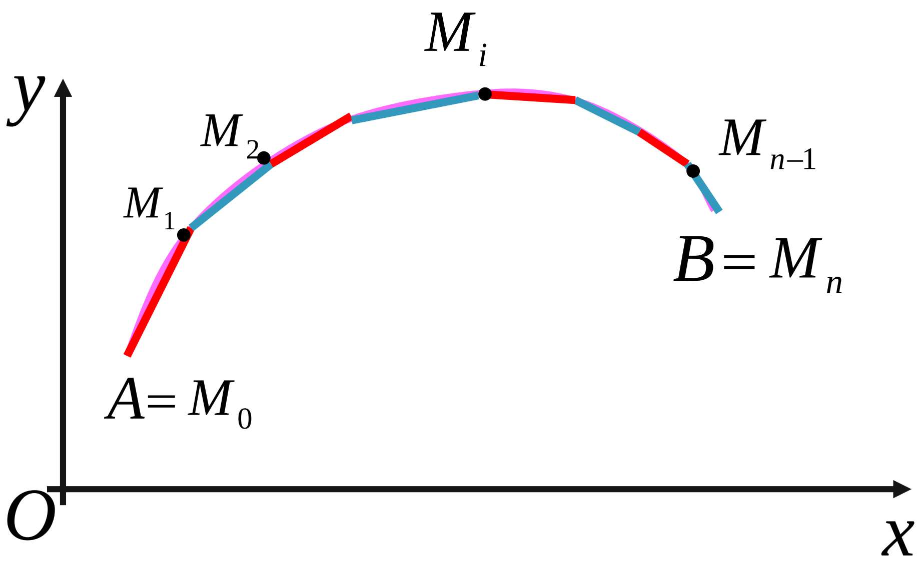
x² − 3x − 4 = 0
4x² − 3x − 1 = 0
∫f(x) dx
m d²x/dt² = −kx
y = mx + b,
Gottfried Wilhelm Leibniz
m d²x/dt² = −kx − f dx/dt + A sin(θt)
and v' = −ky − fv + A sin(wt)
x + b/2a = √(b² − 4ac)/2a
x + b/2a = −√(b² − 4ac)/2a

Calculus(I)

Length of a Plane Curve

Lecturer: Xue Deng

How to compute the arc length of the following figure?



Insert points $A = M_0, M_1, \cdots M_i, \cdots, M_{n-1}, M_n = B$?

If the limit of $\sum_{i=1}^{n} |M_{i-1}M_i|$

then the limit is our arc length of arc AB.

Length of a Plane Curve

Because: $\sqrt{(\mathrm{d}x)^2+(\mathrm{d}y)^2} = \sqrt{1+y'^2}\,\mathrm{d}x$

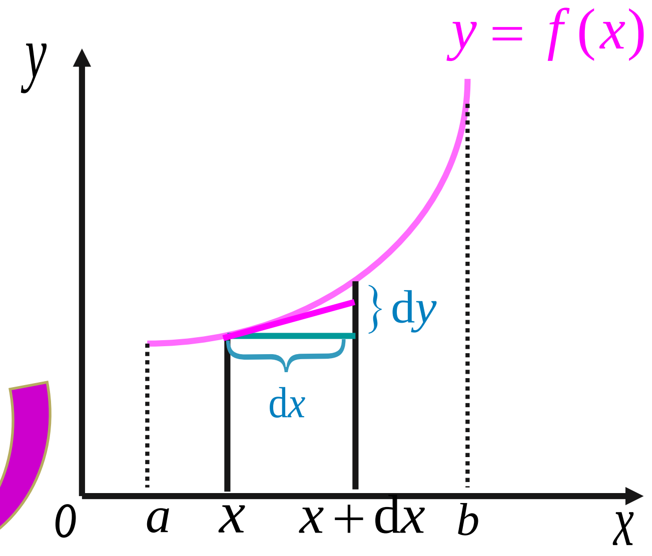
1 Cartesian coordinate system

Arc length element: $\mathrm{d}s=\sqrt{1+y'^2}\,\mathrm{d}x,$

Arc length:

$$s=\int_a^b\sqrt{1+y'^2}\,\mathrm{d}x.$$

$$s=\int_c^d\sqrt{1+x'^2}\,\mathrm{d}y.$$



Length of a Plane Curve

2 Parametric equation system

Curve arc equation: $\begin{cases} x = \phi(t) \\ y = \psi(t) \end{cases} \quad (\alpha \le t \le \beta)$

and $\phi(t), \psi(t)$ has continuous derivative on $[a, b]$.

$$\Rightarrow \quad ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\phi'^2(t) + \psi'^2(t)}\,dt$$

Arc length:

$$s = \int_{\alpha}^{\beta} \sqrt{\phi'^2(t) + \psi'^2(t)}\,dt$$

Length of a Plane Curve

3 Polar coordinate system

Curve arc: $\rho = \rho(\theta)$ $(\alpha \le \theta \le \beta)$ and $\rho(\theta)$

has continuous derivative on $[\alpha, \beta]$.

$$\because \begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} = \begin{cases} x = \rho(\theta)\cos\theta \\ y = \rho(\theta)\sin\theta \end{cases} \quad (\alpha \le \theta \le \beta)$$

$$\therefore \mathrm{d}s = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \sqrt{\rho^2(\theta) + \rho'^2(\theta)}\,\mathrm{d}\theta$$

Arc length:

$$s = \int_{\alpha}^{\beta} \sqrt{\rho^2(\theta) + \rho'^2(\theta)}\,\mathrm{d}\theta.$$

Example 1

Prove the arc length of $y = a\sin x$ $(0 \le x \le 2\pi)$ equal to

the perimeter of $\begin{cases} x = \cos t \\ y = \sqrt{1+a^2}\sin t \end{cases}$ $(0 \le t \le 2\pi)$

$$s_1 = \int_0^{2\pi} \sqrt{1+y'^2}\,dx = \int_0^{2\pi} \sqrt{1+a^2\cos^2 x}\,dx$$

$$= 2\int_0^{\pi} \sqrt{1+a^2\cos^2 x}\,dx$$

elliptical symmetry

$$s_2 = \int_0^{2\pi} \sqrt{(x')^2+(y')^2}\,dt = 2\int_0^{\pi} \sqrt{(\sin t)^2+(1+a^2)(\cos t)^2}\,dt$$

$$= 2\int_0^{\pi} \sqrt{1+a^2\cos^2 x}\,dx = s_1$$

Example 2

Find the cirumference of the circle $x^2 + y^2 = a^2$.

$$\begin{cases} x = a\cos t \\ y = a\sin t \end{cases} (0 \le t \le 2\pi),$$

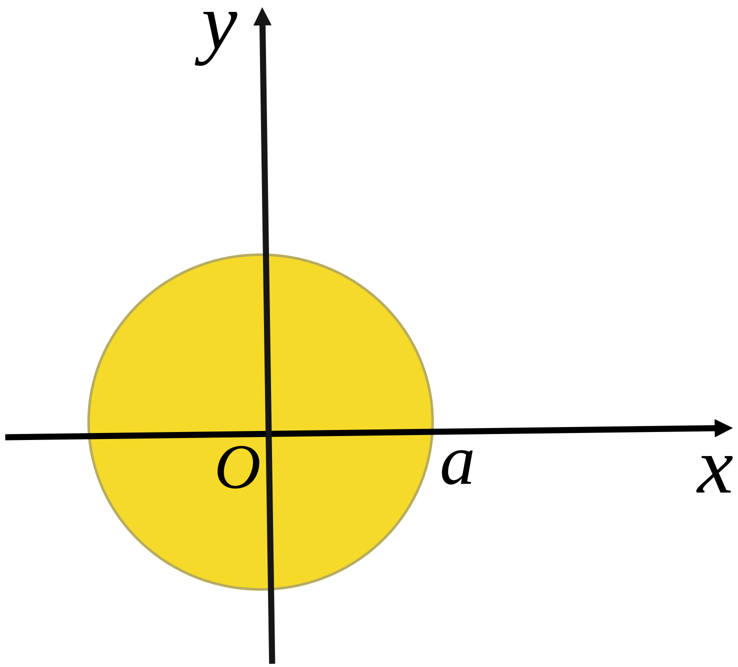
$$\begin{cases} x' = -a\sin t, \\ y' = a\cos t, \end{cases}$$

Polar Situation

so $L = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} dt$

$= \int_0^{2\pi} a dt$

$= 2\pi a.$



Example 3

Find the length of the $\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t), \end{cases} \quad (a > 0).$

$x'(t) = a(1 - \cos t),$ $y'(t) = a \sin t,$

Parametric equation system

$$L = \int_0^{2\pi} \sqrt{(a(1-\cos t))^2 + (a \sin t)^2} \mathrm{d}t$$

$$= a\int_0^{2\pi} \sqrt{2 - 2\cos t} \mathrm{d}t$$

$$= \int_0^{2\pi} 2a \left| \sin \frac{t}{2} \right| \mathrm{d}t$$

$$= \int_0^{2\pi} 2a \sin \frac{t}{2} \mathrm{d}t = 8a$$

Summary of Length of a Plane Curve

1 Cartesian coordinate system

$$s=\int_a^b \sqrt{1+y'^2}\,\mathrm{d}x. \qquad s=\int_c^d \sqrt{1+x'^2}\,\mathrm{d}y.$$

2 Parametric equation system

$$s=\int_\alpha^\beta \sqrt{\phi'^2(t)+\psi'^2(t)}\,\mathrm{d}t$$

3 Polar coordinate system

$$s=\int_\alpha^\beta \sqrt{\rho^2(\theta)+\rho'^2(\theta)}\,\mathrm{d}\theta.$$

A
R
C
L
E
N
G
T
H

Questions and Answers

Q1: Find the length of the line from $A(0,1)$ to $B(5,13)$.

$$y=\frac{12}{5}x+1,\ y'=\frac{12}{5},$$

$$L=\int_0^5\sqrt{1+\left(\frac{12}{5}\right)^2}\,\mathrm{d}x$$

$$=\int_0^5\frac{13}{5}\,\mathrm{d}x$$

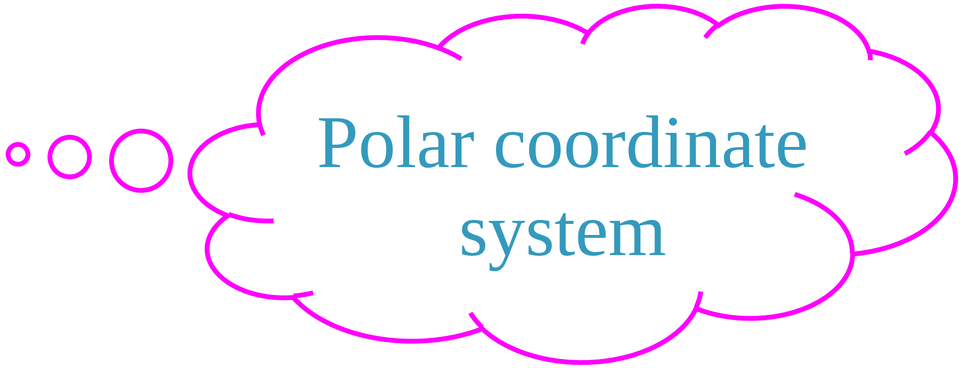
$$=13.$$

Cartesian coordinate system

Questions and Answers

Q2: Find the length of the $r = a\theta$ from $\theta = 0$ to $\theta = 2\pi$. $r'(\theta) = a$

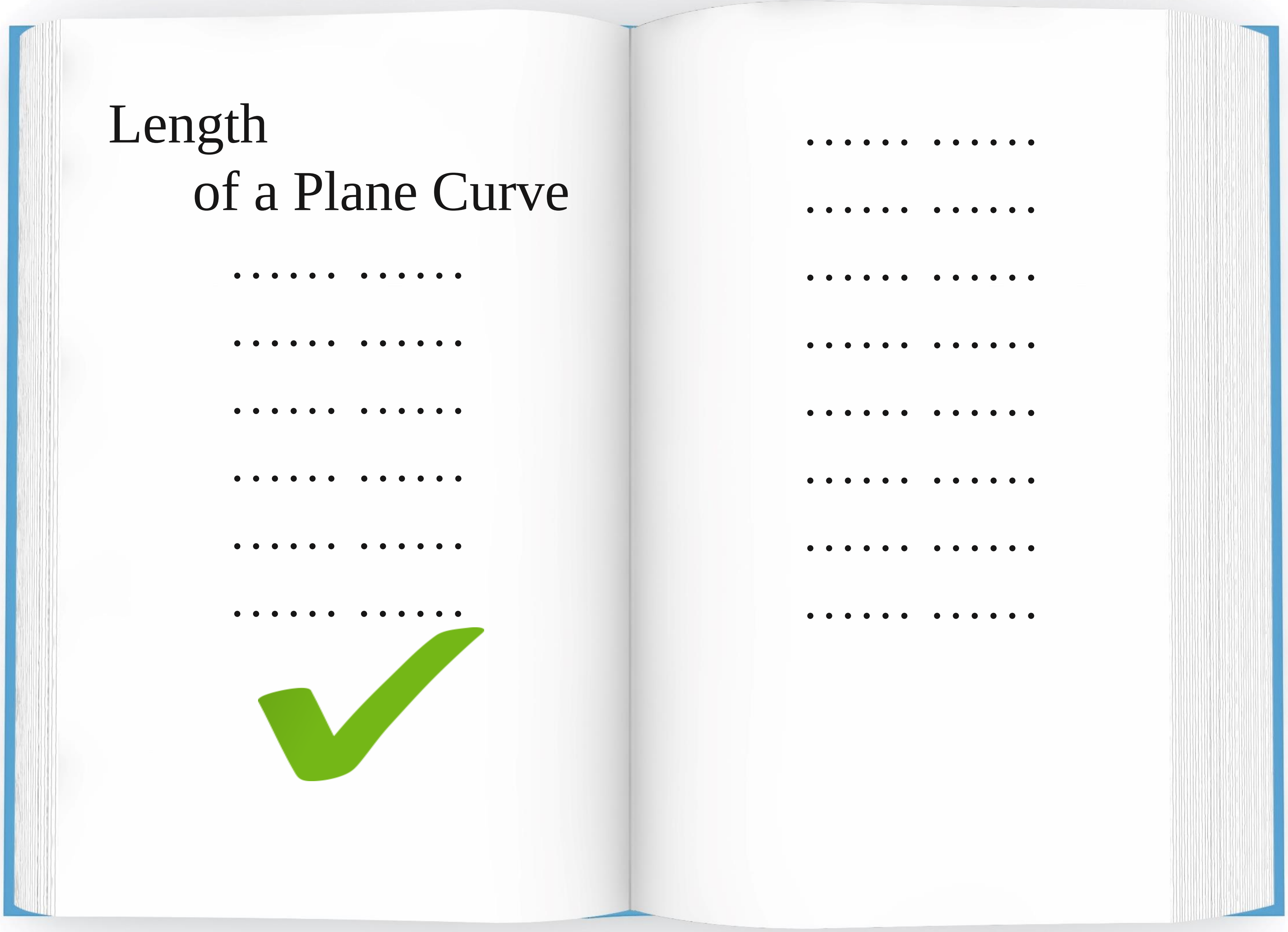
$$L = \int_0^{2\pi} \sqrt{(a\theta)^2 + (a)^2}\,\mathrm{d}\theta$$



$$= a\int_0^{2\pi} \sqrt{1+\theta^2}\,\mathrm{d}\theta$$

$$= a\left[\frac{\theta}{2}\sqrt{1+\theta^2} + \frac{1}{2}\ln(\theta + \sqrt{1+\theta^2})\right]_0^{2\pi}$$

$$= a\left[\pi\sqrt{1+4\pi^2} + \frac{1}{2}\ln(2\pi + \sqrt{1+4\pi^2})\right]$$


Length
of a Plane Curve